On L^1-Convergence of Fourier Series Under $MVBV$ Condition

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Abstract

Let $f \in L_{2\pi}$ be a real-valued even function with its Fourier series $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$, and let $S_n (f, x)$, $n \geq 1$, be the n-th partial sum of the Fourier series. It is well-known that if the nonnegative sequence $\{a_n\}$ is decreasing and $\lim_{n \to \infty} a_n = 0$, then

$$\lim_{n \to \infty} \|f - S_n(f)\|_L = 0 \ \text{ if and only if } \ \lim_{n \to \infty} a_n \log n = 0.$$

We weaken the monotone condition in this classical result to the so-called mean value bounded variation ($MVBV$) condition. The generalization of the above classical result in real-valued function space is presented as a special case of the main result in this paper which gives the L^1-convergence of a function $f \in L_{2\pi}$ in complex space. We also give results on L^1-approximation of a function $f \in L_{2\pi}$ under the $MVBV$ condition.

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1   Introduction

Let $L_{2\pi}$ be the space of all complex-valued integrable functions $f(x)$ of period 2π equipped with the norm

$$\|f\|_L = \int_{-\pi}^{\pi} |f(x)| dx.$$

Denote the Fourier series of $f \in L_{2\pi}$ by

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}, \tag{1}$$

and its partial sum $S_n(f, x)$ by

$$\sum_{k=-n}^{n} \hat{f}(k) e^{ikx}.$$

When $f(x) \in L_{2\pi}$ is a real valued even function, then the Fourier series of f has the form

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \tag{2}$$

correspondingly, its partial sum $S_n(f, x)$ is

$$\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx.$$

The following two classical convergence results can be found in many monographs (see [1] and [9], for example):

Result One: If a nonnegative sequence $\{b_n\}_{n=1}^{\infty}$ is decreasing and $\lim_{n\to\infty} b_n = 0$, then the series $\sum\limits_{n=1}^{\infty} b_n \sin nx$ converges uniformly if and only if $\lim\limits_{n\to\infty} nb_n = 0$.

Result Two: Let $f \in L_{2\pi}$ be an even function and (2) be its Fourier series. If the sequence $\{a_n\}_{n=0}^{\infty}$ is nonnegative, decreasing, and $\lim_{n\to\infty} a_n = 0$, then

$$\lim_{n\to\infty} \|f - S_n(f)\|_L = 0 \text{ if and only if } \lim_{n\to\infty} a_n \log n = 0.$$

These results have been generalized by weakening the monotone conditions of the coefficient sequences. They have also been generalized to the complex valued function spaces. The most recent generalizations of Result One can be found in [8] where the monotonic condition is finally weakened to the $MVBV$ condition (Mean Value Bounded Variation condition, see Corollary 2 in Section 2 for definition), and it is proved to be the weakest possible condition we can have to replace the monotone condition in Result One. The process of generalizing Result Two can be found in many papers, for example, see [2] - [7]. In this paper, we will weaken the monotone condition in Result Two (and all its later generalized conditions, see [8] for the relations between these conditions), to the $MVBV$ condition in the complex valued function spaces (see *Definition 1* in Section 2) in Theorem 1, and give the generalization in real valued function spaces as a special case of Theorem 1 in Corollary 2. Like the important role that the $MVBV$ condition plays in generalizing Result One, although we are not able to prove it here, we propose that Theorem 1 in Section 2 is the ultimate generalization

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of Result Two, i.e. the $MVBV$ condition is also the weakest possible condition we can have to replace the monotone condition in Result Two. We also discuss, under the $MVBV$ condition, the L^1-approximation rate of a function $f \in L_{2\pi}$ in the last section.

Throughout this paper, we always use $C(x)$ to indicate a positive constant depending upon x only, and use C to indicates an absolute positive constant. They may have different values in different occurrences.

2 L^1 convergence

In this section, we first give the definition of $MVBV$ condition, or the class $MVBVS$, and then prove our main result on L^1-convergence of the Fourier series of a complex valued function $f(x) \in L_{2\pi}$ whose coefficients form a sequence in the class $MVBVS$.

Definition 1. Let $\mathbf{c} := \{\mathbf{c_n}\}_{\mathbf{n=0}}^\infty$ be a sequence of complex numbers satisfying $c_n \in K(\theta_1) := \{z : |\arg z| \le \theta_1\}$ for some $\theta_1 \in [0, \pi/2)$ and all $n = 0, 1, 2, \cdots$. If there is a number $\lambda \ge 2$ such that

$$\sum_{k=m}^{2m} |\Delta c_k| := \sum_{k=m}^{2m} |c_{k+1} - c_k| \le C(\mathbf{c}) \frac{1}{\mathbf{m}} \sum_{\mathbf{k=[\lambda^{-1}m]}}^{[\lambda \mathbf{m}]} |\mathbf{c_k}|$$

holds for all $m = 1, 2, \cdots$, then we say that the sequence $\mathbf{c}$ is a Mean Value Bounded Variation Sequence, i.e., $\mathbf{c} \in \mathbf{MVBVS}$, in complex sense, or the sequence $\mathbf{c}$ satisfies the $MVBV$ condition.

Our main result of this paper is:

Theorem 1. Let $f(x) \in L_{2\pi}$ be a complex-valued function. If the Fourier coefficients $\hat{f}(n)$ of f satisfy that $\{\hat{f}(n)\}_{n=0}^{+\infty} \in MVBVS$ and

$$\lim_{\mu \to 1^+} \limsup_{n \to \infty} \sum_{k=n}^{[\mu n]} |\Delta \hat{f}(k) - \Delta \hat{f}(-k)| \log k = 0, \tag{3}$$

where

$$\Delta \hat{f}(k) = \hat{f}(k+1) - \hat{f}(k), \;\; \Delta \hat{f}(-k) = \hat{f}(-k-1) - \hat{f}(-k), \;\; k \ge 0.$$

Then

$$\lim_{n \to \infty} \|f - S_n(f)\|_L = 0$$

if and only if

$$\lim_{n \to \infty} \hat{f}(n) \log |n| = 0.$$

In order to prove Theorem 1, we present the following four lemmas.

Lemma 1. *Let $\{c_n\} \in MVBVS$, then for any given $1 < \mu < 2$, we have*

$$\sum_{k=n}^{[\mu n]} |\Delta c_k| \log k = O\left(\max_{[\lambda^{-1}n] \leq k \leq [\lambda n]} |c_k| \log k\right), \quad n \to \infty,$$

where the implicit constant depends only on the sequence $\{c_n\}$ and λ.

For sufficiently large n, the lemma can be derived directly from the conditions that $1 < \mu < 2$ and $\{c_n\} \in MVBVS$.

Lemma 2. *Let $\{\hat{f}(n)\} \in K(\theta_0)$ for some $\theta_0 \in [0, \pi/2)$, then*

$$\sum_{k=1}^{n} \frac{1}{k} \left|\hat{f}(n+k)\right| = O\left(\|f - S_n(f)\|_L\right)$$

for all $n = 1, 2, \cdots$, where the implicit constant depends only on θ_0.

Proof. Write

$$\phi_{\pm n}(x) := \sum_{k=1}^{n} \frac{1}{k}\left(e^{i(k\mp n)x} - e^{-i(k\pm n)x}\right).$$

It follows from a well-known inequality (e.g. see Theorem 2.5 in [6])

$$\sup_{n \geq 1}\left|\sum_{k=1}^{n} \frac{\sin kx}{k}\right| \leq 3\sqrt{\pi}$$

that

$$|\phi_{\pm n}(x)| \leq 6\sqrt{\pi}.$$

Hence

$$\frac{1}{6\sqrt{\pi}}\left|\int_{-\pi}^{\pi}(f(x) - S_n(f, x))\phi_{\pm n}(x)dx\right| \leq \|f - S_n(f)\|_L,$$

and therefore

$$\left|\sum_{k=1}^{n} \frac{1}{k}\hat{f}(n+k)\right| = O(\|f - S_n(f)\|_L).$$

Now as $\{\hat{f}(n)\} \in K(\theta_0)$ for some $\theta_0 \in [0, \pi/2)$ and for all $n \geq 1$, we have

$$
\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k}\left|\hat{f}(n+k)\right| &\leq C(\theta_0)\sum_{k=1}^{n} \frac{1}{k}\mathrm{Re}\hat{f}(n+k) \\
&\leq C(\theta_0)\left|\sum_{k=1}^{n} \frac{1}{k}\hat{f}(n+k)\right| \\
&= O(\|f - S_n(f)\|_L).
\end{aligned}
$$

Lemma 3. ([5]). *Write*

$$D_k(x) \quad := \quad \frac{\sin((2k+1)x/2)}{2\sin(x/2)},$$

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$$D_k^*(x) \quad : \quad = \begin{cases} \frac{\cos(x/2) - \cos((2k+1)x/2)}{2\sin(x/2)} & |x| \leq 1/n, \\ -\frac{\cos((2k+1)x/2)}{2\sin(x/2)} & 1/n \leq |x| \leq \pi, \end{cases}$$

$$E_k(x) \quad : \quad = D_k(x) + iD_k^*(x).$$

For $k = n, n+1, \cdots, 2n$, we have

$$E_k(\pm x) - E_{k-1}(\pm x) = e^{\pm ikx}, \tag{4}$$

$$E_k(x) + E_k(-x) = 2D_k(x), \tag{5}$$

$$\|E_k\|_L + \|D_k\|_L = O(\log k). \tag{6}$$

Lemma 4. *Let $\{\hat{f}(n)\} \in MVBVS$. If $\lim\limits_{n \to \infty} \|f - S_n(f)\|_L = 0$, then*

$$\lim_{n \to \infty} \hat{f}(n) \log n = 0.$$

Proof. By the definition of $MVBVS$, we derive that for $k = n, n+1, \cdots, 2n$,

$$\begin{aligned}
|\hat{f}(2n)| &\leq \sum_{j=k}^{2n-1} |\Delta \hat{f}(j)| + |\hat{f}(k)| \\
&\leq \sum_{j=k}^{2k} |\Delta \hat{f}(j)| + |\hat{f}(k)| \\
&= O\left(\frac{1}{n} \sum_{j=[\lambda^{-1}k]}^{[\lambda k]} |\hat{f}(j)| \right) + |\hat{f}(k)|.
\end{aligned}$$

Therefore, it follows that from the fact that

$$\log n \leq C(\lambda) \sum_{j=[\lambda]+1}^{[(\lambda+1)^{-2}n]} \frac{1}{j},$$

we have

$$\begin{aligned}
|\hat{f}(2n)| \log n &\leq C(\lambda)|\hat{f}(2n)| \sum_{j=[\lambda]+1}^{[(\lambda+1)^{-2}n]} \frac{1}{j} \\
&\leq C(\lambda) \sum_{j=[\lambda]+1}^{[(\lambda+1)^{-2}n]} \frac{1}{j} \left(\frac{1}{n} \sum_{k=[\lambda^{-1}(n+j)]}^{[\lambda(n+j)]} |\hat{f}(k)| + |\hat{f}(n+j)| \right)
\end{aligned}$$

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$$
= \frac{C(\lambda)}{n} \sum_{j=[\lambda]+1}^{[(\lambda+1)^{-2}n]} \frac{1}{j} \sum_{k=[\lambda^{-1}(n+j)]}^{[\lambda(n+j)]} |\hat{f}(k)|
$$

$$
+ C(\lambda) \sum_{j=1}^{[(\lambda+1)^{-2}n]} \frac{1}{j} |\hat{f}(n+j)|
$$

$$
= : I_1 + I_2. \tag{7}
$$

By applying Lemma 2, we see that

$$
I_2 \leq C(\lambda, \theta_0) \|f - S_n(f)\|_L. \tag{8}
$$

We calculate I_1 as follows (note that we may add more repeated terms in the right hand side of every inequality below):

$$
\begin{aligned}
I_1 \;\leq\; & \frac{C(\lambda)}{n} \sum_{j=[\lambda]+1}^{[(\lambda+1)^{-2}n]} \frac{1}{j} \sum_{k=[\lambda^{-1}n]+[\lambda^{-1}j]}^{[\lambda n]+[\lambda j]+1} |\hat{f}(k)| \\
\;\leq\; & \frac{C(\lambda)}{n} \sum_{j=[\lambda]+1}^{[(\lambda+1)^{-2}n]} \sum_{m=1}^{[(\lambda+1)^2]} \sum_{k=[\lambda^{-1}n]}^{[\lambda n]+1} \frac{\left|\hat{f}\left(m[\lambda^{-1}j]+k\right)\right|}{j} \\
\;\leq\; & \frac{C(\lambda)}{n} \sum_{m=1}^{[(\lambda+1)^2]} \sum_{j=[\lambda]+1}^{[(\lambda+1)^{-2}n]} \sum_{k=0}^{[\lambda n]-[\lambda^{-1}n]+1} \frac{\left|\hat{f}\left([\lambda^{-1}n]+m[\lambda^{-1}j]+k\right)\right|}{j} \\
\;\leq\; & \frac{C(\lambda)}{n} \sum_{m=1}^{[(\lambda+1)^2]} \sum_{k=0}^{[\lambda n]-[\lambda^{-1}n]+1} \sum_{j=1}^{m\left[\left(\lambda(\lambda+1)^2\right)^{-1}n\right]} \frac{\left|\hat{f}\left([\lambda^{-1}n]+k+j\right)\right|}{j} \\
\;\leq\; & \frac{C(\lambda)}{n} \sum_{m=1}^{[(\lambda+1)^2]} \sum_{k=0}^{[\lambda n]-[\lambda^{-1}n]+1} \left\|f - S_{[\lambda^{-1}n]+k}(f)\right\|_L \quad \text{(by Lemma 2)} \\
\;\leq\; & \frac{C(\lambda)}{n} \sum_{k=0}^{[\lambda n]-[\lambda^{-1}n]+1} \left\|f - S_{[\lambda^{-1}n]+k}(f)\right\|_L. \tag{9}
\end{aligned}
$$

Finally, by combining (7) - (9) and the condition

$$
\lim_{n\to\infty} \|f - S_n(f)\|_L = 0,
$$

we get

$$
\lim_{n\to\infty} \hat{f}(2n) \log n = 0.
$$

A similar argument yields that

$$
\lim_{n\to\infty} |\hat{f}(2n+1)| \log n = 0.
$$

This proves Lemma 4.

We now come to the proof of Theorem 1.

Proof of Theorem 1. Sufficiency. Given $\varepsilon > 0$, by (3), there is a $1 < \mu < 2$ such that

$$\sum_{k=n}^{[\mu n]} \left| \Delta \hat{f}(k) - \Delta \hat{f}(-k) \right| \log k \le \varepsilon \tag{10}$$

holds for sufficiently large $n > 0$. Let

$$\tau_{\mu n, n}(f, x) := \frac{1}{[\mu n] - n} \sum_{k=n}^{[\mu n]-1} S_k(f, x)$$

be the Vallée Poussin sum of order n of f. Then we have

$$\lim_{n \to \infty} \| f - \tau_{\mu n, n}(f) \|_L = 0. \tag{11}$$

By (4), (5), and applying Abel transformation, we get

$$
\begin{aligned}
&\tau_{\mu n, n}(f, x) - S_n(f, x) \\
=\ & \frac{1}{[\mu n] - n} \sum_{k=n+1}^{[\mu n]} ([\mu n] - k) \left(\hat{f}(k) e^{ikx} + \hat{f}(-k) e^{-ikx} \right) \\
=\ & \frac{1}{[\mu n] - n} \sum_{k=n}^{[\mu n]} ([\mu n] - k) \left(2 \Delta \hat{f}(k) D_k(x) - (\Delta \hat{f}(k) - \Delta \hat{f}(-k)) E_k(-x) \right) \\
& + \frac{1}{[\mu n] - n} \sum_{k=n}^{[\mu n]-1} \left(\hat{f}(k+1) E_k(x) - \hat{f}(-k-1) E_k(-x) \right) \\
& - \left(\hat{f}(n) E_n(x) + \hat{f}(-n) E_n(-x) \right).
\end{aligned}
\tag{12}
$$

Thus, by (6) and Lemma 1, we have

$$
\begin{aligned}
& \| f - S_n(f) \|_L \\
\le\ & \| f - \tau_{\mu n, n}(f) \|_L + \| \tau_{\mu n, n}(f) - S_n(f) \|_L \\
=\ & \| f - \tau_{\mu n, n}(f) \|_L + O \left(\sum_{k=n}^{[\mu n]} \left| \Delta \hat{f}(k) \right| \log k \right) \\
& + O \left(\sum_{k=n}^{[\mu n]} \left| \Delta \hat{f}(k) - \Delta \hat{f}(-k) \right| \log k \right) \\
& + O \left(\max_{n \le |k| \le [\mu n]} | \hat{f}(k) | \log |k| \right) \\
=\ & \| f - \tau_{\mu n, n}(f) \|_L + O \left(\max_{[\lambda^{-1} n] \le |k| \le [\lambda n]} | \hat{f}(k) | \log |k| \right) \\
& + O \left(\sum_{k=n}^{[\mu n]} \left| \Delta \hat{f}(k) - \Delta \hat{f}(-k) \right| \log k \right),
\end{aligned}
\tag{13}
$$

then
$$\limsup_{n\to\infty} \|f - S_n(f)\|_L \le \varepsilon$$
follows from (10), (11) and the condition that
$$\lim_{n\to\infty} \hat{f}(n) \log |n| = 0.$$
This implies that
$$\lim_{n\to\infty} \|f - S_n(f)\|_L = 0.$$

Necessity. Since $\{\hat{f}(n)\} \in MVBVS$, by applying Lemma 4, we have
$$\lim_{n\to\infty} \hat{f}(n) \log n = 0. \tag{14}$$
In order to prove $\lim_{n\to-\infty} \hat{f}(n) \log |n| = 0$, by applying (12) and (6), we see that for any given μ, $1 < \mu < 2$,

$$
\begin{aligned}
&\|\hat{f}(-n)E_n(-x)\|_L \\
\le\ & \|\tau_{\mu n,n}(f) - S_n(f)\|_L \\
&+ \frac{1}{[\mu n] - n}\left\|\sum_{k=n}^{[\mu n]-1} \hat{f}(-k-1)E_k(-x)\right\|_L \\
&+ O\left(\sum_{k=n}^{[\mu n]}\left(|\Delta\hat{f}(k) - \Delta\hat{f}(-k)|\log k + \left|\Delta\hat{f}(k)\right|\log k\right)\right) \\
&+ O\left(\max_{n\le k\le[\mu n]}|\hat{f}(k)|\log k\right). \tag{15}
\end{aligned}
$$

It is not difficult to see that
$$\left\|\sum_{k=n}^{[\mu n]-1}\hat{f}(-k-1)E_k(-x)\right\|_L = I + O\left(n\max_{n<k\le[\mu n]}|\hat{f}(-k)|\right),$$
where
$$I := \int_{n^{-1}\le|x|\le\pi}\left|\frac{1}{2\sin(x/2)}\sum_{k=n}^{[\mu n]-1}\hat{f}(-k-1)e^{\frac{i(2k+1)x}{2}}\right|dx.$$
Since the trigonometric function system is orthonormal, we have

$$
\begin{aligned}
I\ \le\ & \left(\int_{n^{-1}\le|x|\le\pi}\left|\sum_{k=n}^{[\mu n]-1}\hat{f}(-k-1)e^{\frac{i(2k+1)x}{2}}\right|^2 dx\right)^{1/2} \\
& \times\left(\int_{n^{-1}}^{\pi}\frac{1}{\sin^2(x/2)}dx\right)^{1/2} \\
=\ & O\left(\sqrt{n}\left(\sum_{k=n+1}^{[\mu n]}|\hat{f}(-k)|^2\right)^{1/2}\right) \\
=\ & O\left(n\max_{n\le k\le[\mu n]}|\hat{f}(-k)|\right),
\end{aligned}
$$

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which yields that

$$\frac{1}{[\mu n] - n} \left\| \sum_{k=n}^{[\mu n]-1} \hat{f}(-k-1)E_k(-x) \right\|_L = O\left(\max_{n < k \leq [\mu n]} |\hat{f}(-k)| \right). \tag{16}$$

By combining (11), (14) - (16), with Lemma 1 and the condition

$$\lim_{n \to \infty} \|f - S_n(f)\|_L = 0,$$

and the fact (since $f \in L_{2\pi}$) that

$$\lim_{n \to \infty} \hat{f}(-n) = 0,$$

we have for $n \to \infty$,

$$\begin{aligned}
\|\hat{f}(-n)E_n(-x)\|_L &\leq \sum_{k=n}^{[\mu n]} |\Delta \hat{f}(k) - \Delta \hat{f}(-k)| \log k \\
&\quad + \|\tau_{\mu n, n}(f) - S_n(f)\|_L \\
&\quad + O\left(\max_{[\lambda^{-1} n] \leq k \leq [\lambda n]} |\hat{f}(k)| \log k \right) \\
&\quad + O\left(\max_{n < k \leq [\mu n]} |\hat{f}(-k)| \right) \\
&= \sum_{k=n}^{[\mu n]} |\Delta \hat{f}(k) - \Delta \hat{f}(-k)| \log k + o(1). \tag{17}
\end{aligned}$$

On the other hand, we have

$$\|\hat{f}(-n)E_n(-x)\|_L \geq |\hat{f}(-n)| \|D_n(x)\|_L \geq \frac{1}{\pi} |\hat{f}(-n)| \log n. \tag{18}$$

Hence, from (17), (18), and (10), we have that

$$|\hat{f}(-n)| \log n \leq \sum_{k=n}^{[\mu n]} |\Delta \hat{f}(k) - \Delta \hat{f}(-k)| \log k \leq \varepsilon$$

holds for sufficiently large n, which, together with (14), completes the proof of necessity.

In view of Lemma 1, we can see that the condition (3) in Theorem 1 can be replaced by the following condition

$$\lim_{\mu \to 1^+} \limsup_{n \to \infty} \sum_{k=n}^{[\mu n]} |\Delta \hat{f}(-k)| \log k = 0,$$

and the proof of the result is easier. Therefore we have a corollary to Theorem 1.

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Corollary 1. *Let $f(x) \in L_{2\pi}$ be a complex valued function. If both $\{\hat{f}(n)\}_{n=0}^{+\infty} \in MVBVS$ and $\{\hat{f}(-n)\}_{n=0}^{+\infty} \in MVBVS$, then*

$$\lim_{n \to \infty} \|f - S_n(f)\|_L = 0$$

if and only if

$$\lim_{n \to \infty} \hat{f}(n) \log |n| = 0.$$

If $f(x)$ is a real valued function, then its Fourier coefficients $\hat{f}(n)$ and $\hat{f}(-n)$ are a pair of conjugate complex numbers. Consequently, $\{\hat{f}(n)\}_{n=0}^{+\infty} \in MVBVS$ if and only if $\{\hat{f}(-n)\}_{n=0}^{+\infty} \in MVBVS$. Thus, we have the following generalization of the classical result (cf. Result Two in the introduction):

Corollary 2. *Let $f(x) \in L_{2\pi}$ be a real valued even function and (2) be its Fourier series. If $\mathbf{A} = \{a_n\}_{n=0}^{+\infty} \in MVBVS$ in real sense, i.e. $\{a_n\}$ is a nonnegative sequence, and there is a number $\lambda \geq 2$ such that*

$$\sum_{k=m}^{2m} |\Delta a_k| \leq C(\mathbf{A}) \frac{1}{\mathbf{m}} \sum_{\mathbf{k}=[\lambda^{-1}\mathbf{m}]}^{[\lambda\mathbf{m}]} \mathbf{a_k}$$

for all $n = 1, 2, \ldots$, then

$$\lim_{n \to \infty} \|f - S_n(f)\|_L = 0$$

if and only if

$$\lim_{n \to \infty} a_n \log n = 0.$$

3 L^1 Approximation

Let $E_n(f)_L$ be the best approximation of a complex valued function $f \in L_{2\pi}$ by trigonometric polynomials of degree n in L^1 norm, that is,

$$E_n(f)_L := \inf_{c_k} \left\| f - \sum_{k=-n}^{n} c_k e^{ikx} \right\|_L.$$

We establish the corresponding L^1-approximation theorem in a similar way to Theorem 1:

Theorem 2. *Let $f(x) \in L_{2\pi}$ be a complex valued function, $\{\psi_n\}$ a decreasing sequence tending to zero with*

$$\psi_n \sim \psi_{2n}, \tag{19}$$

i.e., there exist positive constants C_1 and C_2, such that $C_1\psi_n \leq \psi_{2n} \leq C_2\psi_n$. If both $\{\hat{f}(n)\}_{n=0}^{+\infty} \in MVBVS$ and $\{\hat{f}(-n)\}_{n=0}^{+\infty} \in MVBVS$, then

$$\|f - S_n(f)\|_L = O(\psi_n) \tag{20}$$

if and only if

$$E_n(f)_L = O(\psi_n) \quad and \quad \hat{f}(n)\log|n| = O(\psi_{|n|}). \tag{21}$$

Proof. Under the condition of Theorem 2, we see from (13) in the proof of Theorem 1 that

$$\begin{aligned}
\|f - S_n(f)\|_L &\leq \|f - \tau_{\mu n, n}(f)\|_L + O\left(\max_{[\lambda^{-1}n]\leq|k|\leq[\lambda n]} |\hat{f}(k)|\log|k|\right) \\
&\leq C(\mu)E_n(f) + O\left(\max_{[\lambda^{-1}n]\leq|k|\leq[\lambda n]} |\hat{f}(k)|\log|k|\right),
\end{aligned}$$

thus (20) holds if (19) and (21) hold. Now if (20) holds, then

$$E_n(f)_L = O(\psi_n)$$

and

$$\|f - \tau_{\mu n, n}(f)\|_L = O(\psi_n).$$

From (7) - (9) in the proof of Lemma 4 and condition (19), we have

$$\begin{aligned}
|\hat{f}(n)|\log n &\leq \frac{C(\lambda)}{n} \sum_{j=1}^{[\lambda n]-[\lambda^{-1}n]+1} \left\|f - S_{[\lambda^{-1}n]+j}(f)\right\|_L \\
&\quad + C(\lambda)\|f - S_n(f)\|_L \\
&= O(\psi_n).
\end{aligned} \tag{22}$$

Since $\{\hat{f}(-n)\}_{n=0}^{+\infty} \in MVBVS$, by a similar argument to (22), we also have

$$\left|\hat{f}(-n)\right|\log n = O(\psi_n).$$

This completes the proof of Theorem 2.

In particular, if we take

$$\psi_n := \frac{1}{(n+1)^r}\omega\left(f^{(r)}, \frac{1}{n+1}\right)_L,$$

where r is a positive integer, and $\omega(f,t)_L$ is the modulus of continuity of f in L^1 norm, i.e.

$$\omega(f,t)_L := \max_{0\leq h\leq t} \|f(x+h) - f(x)\|_L.$$

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By Theorem 2 and the Jackson theorem (e.g. see [6] or [9]) in L^1−space, we immediately have

Corollary 3. *Let $f(x) \in L_{2\pi}$ be a complex valued function. If both $\{\hat{f}(n)\}_{n=0}^{+\infty} \in MVBVS$ and $\{\hat{f}(-n)\}_{n=0}^{+\infty} \in MVBVS$ hold, then*

$$\|f - S_n(f)\|_L = O\left(\frac{1}{(n+1)^r}\omega\left(f^{(r)}, \frac{1}{n+1}\right)_L\right)$$

if and only if

$$\hat{f}(n)\log|n| = O\left(\frac{1}{(n+1)^r}\omega\left(f^{(r)}, \frac{1}{n+1}\right)_L\right).$$

This corollary generalizes the corresponding results in [5] and [2].

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